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A simple nonparametric test for structural change in joint tail probabilities

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A simple nonparametric test for structural change in joint tail probabilities[1]

by

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Summary

We propose a new test against a change in the probability of multivariate tail events. The test is based on partial sums of a suitably defined indicator function and detects multiple changes in joint tail probabilities better than a previously suggested competitor.

Keywords: stock returns, copulas, structural change.

JEL numbers: C12, C14, G1

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1 Introduction and Summary

In 2008, all major stock markets in the world fell by roughly 30% to 40%. The exact figures are: DJIA -33.8%, S&P 500 -38.5%, CAC40 -42.6%, FTSE 100 -31.3% and DAX -40.2% (source: finance.yahoo). The year before, there were likewise some extreme events, with China for instance growing by 97%, but the dependence among markets was much weaker. Even when focussing on downturns in the market, the joint behavior of stocks in 2008 appears to be unique in recent history. For instance, even though Germany went down even further in 2002 (by -44%), this downturn was not then shared by others to the same extent. The question therefore arises whether this can be explained by chance or whether there was a structural change in joint tail probabilities sometime in between.

It is important to distinguish two issues here. The first is a possible asymmetry of dependence in the upper and lower tails of a joint distribution. It has by now been firmly established that joint stock returns exhibit larger dependence in the lower than in the upper tail (Ang & Chen (2002), Fortin & Kuzmicz (2002), Vaz de Melo Mendez (2005), Sun et al. (2008), among many others). This implies that joint downside moves are more probable than joint upside moves even when there is no structural change at all. The second issue is a change in the dependence structure itself, as for instance investigated by Campbell et al. (2002, 2008) and Forbes & Rigobon (2002). And it is this problem which we address in the present paper.

Following Busetti & Harvey (2008), we base our test on joint exceedances of certain quantiles of the marginal distributions. Instead of using sums of squares of a normalized indicator function, we propose two alternative test statistics. The first is based on the maximum of cumulative sums of the indicator variables, in the spirit of Ploberger & Krämer (1992). The second uses the range of the cumulative sums. We show via Monte Carlo simulation that no test is uniformly superior to the others. While the sums of squares version is more likely to detect gradual or continuous changes in probabilities, the max test and the range test are more successful with abrupt changes. None of the tests requires prior knowledge as to when a structural change occurs.

2 The test and its asymptotic null distribution

Following Busetti & Harvey (2008), we let $\xi(\tau)$ denote the τ-quantile of some univariate probability distribution. To avoid unnecessary notational complications, we consider continuous distributions only, so $\xi(\tau)$ is uniquely defined.

For a bivariate series y_{1t} and y_{2t}, $t = 1, \ldots, T$, let $\tilde{\xi}(\tau_1)$ and $\tilde{\xi}(\tau_2)$ denote the respective empirical quantiles, and let $C_T(\tau_1, \tau_2)$ be the proportion of observation where both y_{1t} and y_{2t} are less than or equal to $\tilde{\xi}(\tau_1)$ or $\tilde{\xi}(\tau_2)$, respectively. $C_T(\tau_1, \tau_2)$ is an estimator of $p := P(y_{1t} \leq \xi(\tau_1), y_{2t} \leq \xi(\tau_2))$, and this parameter is assumed to remain constant under our null hypothesis. For simplicity, we let $\tau_1 = \tau_2 = \tau$ from now on. Without loss of generality, we also confine ourselves to the bivariate case in what follows. Generalizations to higher dimensions are obvious.

In applications, one is usually interested mostly in negative tail events, so typical values of τ are 1% or 5%. Of course, the whole analysis extends to positive tail events, by reversing the inequality signs, and by taking $\tau = 95\%$ or $\tau = 99\%$, and even more generally, to any changes in the copula of y_1 and y_2. In fact, our test may be viewed as a procedure to check the constancy of a copula at a given point. An important application is the pricing of rainbow options where the probabilities of joint exceedances of multivariate returns are a crucial input of the pricing function.

The basic input of our test is what Busetti & Harvey (2008) call the τ-*biquantic*, defined as

$$BIQ_\tau(t) = C_T(\tau, \tau) - I(y_{1t} \leq \tilde{\xi}_1(\tau), y_{2t} \leq \tilde{\xi}_2(\tau)), \; t = 1, \ldots, T, \qquad (1)$$

where $I(.)$ is the indicator function of the event in parentheses. By definition, the $BIQ_\tau(t)$ add to zero, and their partial sums should not deviate too much from zero if $P(y_{1t} \leq \tilde{\xi}(\tau), y_{2t} \leq \tilde{\xi}(\tau))$ remains constant across the sample. On the other hand, if this probability changes at $t = t^*$, say, then the $BIQ_\tau(t)$ will tend to be positive up to t^* when the probability decreases, and the $BIQ_\tau(t)$ will tend to be negative up to t^* when the probability increases. In both cases, the cumulated sum of the $BIQ_\tau(t)$ will move away from zero farther than can be expected under the null hypothesis. This motivates our choice of test statistic, which is a suitably normalized version of

$$B_\tau(T) := \max_{t=1,\ldots,T} \left| \sum_{i=1}^{t} BIQ_\tau(i) \right|. \tag{2}$$

We show below that, under the null and whenever the events $(y_{1t} \leq \xi(\tau), y_{2t} \leq \xi(\tau))$ and $(y_{1s} \leq \xi(\tau), y_{2s} \leq \xi(\tau))$ are independent for all $t \neq s$, the stochastic process

$$B_T(s) := \frac{1}{\sqrt{TC_T(\tau,\tau)(1 - C_T(\tau,\tau))}} \left[\sum_{i=1}^{Ts} BIQ_\tau(i) \right] \quad (0 \leq s \leq 1) \tag{3}$$

tends in distribution to a Brownian Bridge as $T \to \infty$, so the limiting null distribution of

$$\frac{1}{\sqrt{TC_T(\tau,\tau)(1 - C_T(\tau,\tau))}} B_\tau(T) \tag{4}$$

is identical to that of the Kolmogorov-Smirnov test (see Ploberger & Krämer (1992)). Some useful critical values are 1,22 (α=10%), 1,36 (α=5%) and 1,63 (α=1%), where α denotes the significance level. The assumption of independence can be relaxed, as is shown below.

An alternative test statistic can be derived by examining the range of the $BIQ_\tau(t)$, as in Krämer & Schotman (1992). In that case, the test statistic is

$$\frac{1}{\sqrt{TC_T(\tau,\tau)(1 - C_T(\tau,\tau))}} \left[\max_{t=1,\ldots,T} \sum_{i=1}^{t} BIQ_\tau(i) - \min_{t=1,\ldots,T} \sum_{i=1}^{t} BIQ_\tau(i) \right],$$

where the asymptotic null distribution is given by

$$P(X \leq x) = 1 + 2 \sum_{k=1}^{\infty} \left(1 - 4k^2x^2\right) \exp(-2(kx)^2)$$

(see e.g. Kennedy (1976)). Some useful critical values are 1.620 ($\alpha = 10\%$), 1.747 ($\alpha = 5\%$) and 2.001 ($\alpha = 1\%$).

Of course, other functionals of the $BIQ_\tau(t)$ such as the sum of absolute values might also be used as test statistics, but we focus here on the performance of the *maximum* and the *range* statistic (as compared to the sum of *squares* statistic proposed by Busetti & Harvey (2008)).

The convergence in distribution to a Brownian Bridge of $B_\tau(s)$ is probably best seen by first considering

$$\widetilde{BIQ}_\tau(t) = p - I\left(y_{1t} < \xi_1(\tau), y_{2t} < \xi_2(\tau)\right). \tag{5}$$

This is an i.i.d. sequence with zero expectation, finite higher moments of all orders and variance $\sigma^2 = p(1-p)$, so, by standard results from probability theory (see e.g. Billingsley (1986))

$$\widetilde{B}_T(s) := \frac{1}{\sqrt{T\sigma^2}} \sum_{i=1}^{Ts} \widetilde{BIQ}_\tau(i) \tag{6}$$

tends in distribution to a standard Wiener Process and

$$B_T^*(s) := \widetilde{B}_T(s) - \frac{1}{\sqrt{T\sigma^2}} \frac{[Ts]}{T} \sum_{i=1}^{T} \widetilde{BIQ}_\tau(i) \tag{7}$$

tends in distribution to a Brownian Bridge. The convergence to a Brownian Bridge of $B_T(s)$ then follows from the fact that $C_T(\tau,\tau)(1 - C_T(\tau,\tau))$ is consistent for $\sigma^2 = p(1-p)$ and

$$\max_{t=1,\ldots,T} \left| \sum_{i=1}^{t} BIQ_\tau(i) - \sum_{i=1}^{t} \widetilde{BIQ}_\tau(i) \right| \xrightarrow{p} 0 \tag{8}$$

as $T \to \infty$.

In empirical applications, where y_{1t} and y_{2t} are for instance stock returns, the events $(y_{1t} \le \xi(\tau), y_{2t} \le \xi(\tau))$ and $(y_{1s} \le \xi(\tau), y_{2s} \le \xi(\tau))$ are in general not independent. In particular, given that $(y_{1t} \le \xi(\tau), y_{2t} \le \xi(\tau))$ has occurred, with τ in the range of $1\% - 5\%$, the conditional probability of $(y_{1,t+1} \le \xi(\tau), y_{2,t+1} \le \xi(\tau))$ will in general be larger than its unconditional probability due to the well known GARCH-effect.

One way to get around this problem is to replace the variance estimator

$$\hat{\sigma}^2 = C_T(\tau,\tau)(1 - C_T(\tau,\tau)) \tag{9}$$

by some autocorrelation-consistent version (see e.g. Busetti & Harvey (2008, p.13)) and to invoke the weak dependence of the joint tail events to show that $B_T(s)$, properly adjusted, still tends to a Brownian Bridge. Another possibility is to first fit a GARCH-model to y_{1t} and y_{2t} separately and then apply the tests to the standardized empirical innovations (residuals) obtained from these models.

3 Some finite sample Monte Carlo evidence

Following Busetti & Harvey (2008), we examine the performance of the proposed tests using simulated values from the Gaussian and the Clayton copula. We explicitly analyze the effect of multiple breaks in the copula parameter. The results in this and the subsequent sections are generated using Ox (see Doornik (2005)) and the G@ARCH package of Laurent & Peters (2006).

Suppose that there are m breakpoints denoted by $t_1, \ldots, t_m$. Let θ_j denote copula parameter on segment $j = 1, \ldots, m+1$. The bivariate time series y_{1t} and y_{2t}, $t = t_{j-1}+1, \ldots, t_j$ are drawn from a bivariate Gaussian distribution with correlation θ_j or a Clayton copula $C(u, v; \theta_j)$ with dependence parameter θ_j.

In our base case scenario, we simulate 50000 replications of length 3000, and restrict the number of copula parameters such that $\theta_1 \equiv \theta_{2k+1}$ and $\theta_2 \equiv \theta_{2k}$, $k = 0, 1, \ldots$. Intuitively, the series consist of alternating periods of low dependence and periods of high dependence. Finally, we apply the tests to the 0.05, 0.1, 0.25 and 0.50 quantiles, respectively.

In the Gaussian simulation, θ_1 equals 0.5 and θ_2 takes respectively the values 0.1, 0.25, 0.5, 0.75 and 0.9. Table 1 shows that the squares test outperforms the maximum and range tests if there is a single break in the copula parameter. However, the power of our test statistics is higher in the case of two structural breaks, and the test based on the range outperforms the test based on the maximum. Notice also that the maximum and range tests are conservative.

We also examined the sensitivity of our results by reducing the number of observations and by altering the time point of the structural break. For smaller samples the range statistic retains some power if the break in the correlation parameter is sufficiently high. Reducing the length of the second segment also reduces the power of the test statistics. Finally, all tests have low power at the 1% quantile. Explicit results are available from the authors upon request.

The Clayton copula takes the values $\theta_1 = 1$ and $\theta_2 = 1$, 2.5, 7.5 and 15, respectively. Table 2 shows the simulation results. Again the maximum and range outperform the squares test if there are multiple breaks in the series. The results of the sensitivity analysis are in line with the results of the Gaussian simulation.

4 An application to stock returns

We illustrate our tests for the Kuala Lumpur stock exchange in Malaysia and the Hang-Seng index in Hong-Kong. The data has been obtained from Econstats and consists of daily observations from December, 7, 1993 through May, 18, 2009. The corresponding return series is calculated as $y_t = log(x_t/x_{t-1}) \times 100$, where x_t denotes the index at time $t = 1, \ldots, T$. In the analysis below we only keep the dates at which both return series are observed.

For each series we estimated an AR(1)-GARCH(1,1) model with Student-t innovations, so the model becomes

$$\begin{aligned} y_t &= \mu + \phi y_{t-1} + \varepsilon_t \\ \varepsilon_t &= z_t \sigma_t \\ \sigma_t^2 &= \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \end{aligned}$$

where z_t is Student-t with υ degrees of freedom (to be estimated from the data).

Subsequently, the different tests are applied to the standardized empirical innovation series. Since the standardized residuals still contain serial correlation, we replaced the variance of the BIC series by a long-run estimator with 36 lags. The number of lags is based on the bandwidth rule $b = 4(T/100)^{1/4}$. Table 3 shows that only the range-test is able to reject the null-hypothesis for the 0.1 quantile (using a 5% significance level). Furthermore, only the squares test does not reject H_0 for the 0.75 quantile. Finally, all tests do reject the null hypothesis for the median.

To examine the stability of our results we also calculated the results using $b = 0$ (i.e. standard variance of BIC series). In that case only the range test rejects the null hypothesis at the 0.05 and 0.1 quantile, while only the square test is *not* able to reject the null hypothesis at the 0.25 quantile. Finally, all tests reject the null hypothesis for the median and the 0.75 quantile.

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Table 1: Gaussian Copula with structural breaks in correlation: simulated rejection frequencies (T=3000, Rep = 50000, $\theta_1 = 0.5$)

m	test	τ	θ_2	0.10	0.25	0.50	0.75	0.90
1	Squares	0.05		0.68	0.35	0.05	0.46	0.93
		0.10		0.90	0.53	0.05	0.65	0.99
		0.25		0.98	0.71	0.05	0.80	1.00
		0.50		0.95	0.63	0.05	0.76	1.00
1	Maximum	0.05		0.65	0.31	0.04	0.44	0.93
		0.10		0.90	0.51	0.04	0.64	0.99
		0.25		0.98	0.71	0.05	0.80	1.00
		0.50		0.96	0.63	0.05	0.76	1.00
1	Range	0.05		0.48	0.20	0.03	0.31	0.84
		0.10		0.80	0.37	0.04	0.50	0.97
		0.25		0.95	0.57	0.04	0.68	1.00
		0.50		0.91	0.49	0.04	0.63	1.00
2	Squares	0.05		0.14	0.09	0.05	0.07	0.32
		0.10		0.23	0.11	0.05	0.13	0.57
		0.25		0.44	0.15	0.05	0.20	0.79
		0.50		0.37	0.13	0.05	0.18	0.74
2	Maximum	0.05		0.19	0.11	0.04	0.11	0.39
		0.10		0.34	0.16	0.05	0.19	0.64
		0.25		0.56	0.23	0.05	0.28	0.84
		0.50		0.49	0.20	0.05	0.26	0.81
2	Range	0.05		0.31	0.14	0.04	0.27	0.78
		0.10		0.64	0.27	0.04	0.42	0.93
		0.25		0.89	0.45	0.04	0.59	0.99
		0.50		0.82	0.40	0.04	0.53	0.98
3	Squares	0.05		0.19	0.10	0.05	0.13	0.36
		0.10		0.33	0.15	0.05	0.19	0.55
		0.25		0.52	0.22	0.05	0.26	0.75
		0.50		0.43	0.19	0.05	0.24	0.72
3	Maximum	0.05		0.16	0.09	0.04	0.12	0.38
		0.10		0.35	0.15	0.05	0.19	0.61
		0.25		0.58	0.22	0.05	0.28	0.81
		0.50		0.49	0.20	0.05	0.26	0.78
3	Range	0.05		0.12	0.06	0.03	0.09	0.29
		0.10		0.26	0.10	0.04	0.14	0.53
		0.25		0.50	0.17	0.04	0.22	0.78
		0.50		0.41	0.15	0.04	0.20	0.75

Table 2: Clayton Copula with structural breaks in dependence: simulated rejection frequencies (T=3000, Rep = 50000, $\theta_1 = 1$)

m	test	τ	θ_2	1	2.5	7.5	15
		0.05		0.05	0.41	0.76	0.83
1	Squares	0.10		0.05	0.65	0.96	0.98
		0.25		0.05	0.89	1.00	1.00
		0.50		0.05	0.87	1.00	1.00
		0.05		0.04	0.39	0.75	0.83
1	Maximum	0.10		0.04	0.64	0.96	0.99
		0.25		0.05	0.89	1.00	1.00
		0.50		0.05	0.87	1.00	1.00
		0.05		0.04	0.27	0.62	0.71
1	Range	0.10		0.04	0.50	0.91	0.96
		0.25		0.05	0.80	1.00	1.00
		0.50		0.05	0.78	1.00	1.00
		0.05		0.05	0.07	0.17	0.20
2	Squares	0.10		0.05	0.13	0.42	0.53
		0.25		0.05	0.27	0.89	0.96
		0.50		0.05	0.25	0.97	1.00
		0.05		0.04	0.11	0.24	0.28
2	Maximum	0.10		0.04	0.19	0.51	0.61
		0.25		0.05	0.37	0.92	0.97
		0.50		0.05	0.35	0.98	1.00
		0.05		0.04	0.22	0.54	0.62
2	Range	0.10		0.04	0.42	0.86	0.92
		0.25		0.04	0.71	1.00	1.00
		0.50		0.05	0.68	1.00	1.00
		0.05		0.05	0.12	0.24	0.27
3	Squares	0.10		0.05	0.19	0.45	0.53
		0.25		0.05	0.34	0.84	0.92
		0.50		0.05	0.32	0.94	0.99
		0.05		0.04	0.11	0.24	0.29
3	Maximum	0.10		0.04	0.19	0.49	0.59
		0.25		0.05	0.37	0.89	0.95
		0.50		0.05	0.35	0.96	1.00
		0.05		0.04	0.08	0.18	0.22
3	Range	0.10		0.04	0.15	0.41	0.51
		0.25		0.04	0.30	0.87	0.95
		0.50		0.05	0.28	0.97	1.00

Table 3: Test statistics based on standardized innovations of the AR(1)-GARCH-t(1,1) model.

τ	standard Squares	 Maximum	 Range	variance (36 lags) Squares	 Maximum	 Range
0.05	0.3255	$1.4080^{(b)}$	$2.1965^{(a)}$	0.1878	1.0693	$1.6681^{(c)}$
0.10	$0.3884^{(c)}$	$1.5605^{(b)}$	$2.3080^{(a)}$	0.2495	$1.2507^{(c)}$	$1.8498^{(b)}$
0.25	0.2678	$1.4002^{(b)}$	$1.8626^{(b)}$	0.1588	1.0782	1.4342
0.50	$0.4280^{(c)}$	$1.3964^{(b)}$	$2.0978^{(a)}$	0.3358	$1.2369^{(c)}$	$1.8582^{(b)}$
0.75	0.0683	0.6114	1.1386	0.0589	0.5676	1.0570
0.90	0.1285	0.8411	1.1594	0.1154	0.7970	1.0987
0.95	0.0564	0.6303	1.1782	0.0587	0.6432	1.2024

Table shows test statistics using standard variance estimate of BIC series and a long run variance estimate based on 36 lags. Significance is denoted by the superscripts 1% (^{a}), 5% (^{b}) and 10% (^{c}).